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Fixed-Pitch Propeller Selection for Light Airplanes

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A numerical comparison is made between fixed-pitch propellers that either have a constant angle of attack with varying pitch or have a constant pitch ratio that has the angle of attack increasing from tip to hub. It is shown that the constant angle of attack is a better choice for low-speed light airplanes. New equations are derived for more easily determining the variation in blade angles of attack when either the flight velocity or the propeller's rotational speed are changed from the design advance ratio. These equations are useful for selecting the best propeller pitch ratio for a desired performance. Finally, it is shown how the propeller's revolutions per minute can be varied to maintain the desirable constant angle of attack over a wider speed range.

Nomenclature

C_1 = $\frac{3}{4}\tan(\phi_{3/4} + 6\ \text{deg}) = (p/2\pi r_0)$ = pitch ratio for Table 1
C_2 = $(1+\delta)V/r_0\omega$ = aerodynamic advance ratio = $C_2(V/\omega)$
k = $C_2(V/\omega)/C_2(V_p/\omega_p) = [(1+\delta)V/(1+\delta_p)V_p](\omega_p/\omega)$
p = $2\pi r_0(\frac{3}{4}\tan\beta_{3/4})$ = propeller pitch
α = $(\beta - \phi)$ = angle of attack (Fig. 1)
β = blade-pitch angle measured from zero lift (Fig. 1)
β_f = blade-pitch angle measured from flat lower surface
ϕ = angle of resultant airflow = $\tan^{-1}(1+\delta)V/r\omega$ (Fig. 1)
χ = r/r_0, where r_0 = tip radius of propeller

Subscripts

p = design advance ratio, $(V_p/r_0\omega_p) = C_2(V_p/\omega_p)/(1+\delta_p)$
χ = r/r_0 = value for any blade section
$*$ = conditions for maximum lift-drag ratio

I. Introduction

A SIMPLIFIED procedure is presented to select the pitch ratio of a fixed-pitch propeller more easily that would have the best performance at a desired flight velocity. von Mises[1] has shown that a constant angle of attack can be attained along the blade of a fixed-pitch propeller with blade angles measured from zero lift, only if the pitch ratio is appropriately varied along the blade. The present study compares the performance of a constant-pitch propeller that has a constant angle of attack along the blade at a given flight velocity, with that of the commonly used constant pitch propeller for a typical light airplane. New relations are developed to show how the blade angles of a fixed-pitch propeller can be selected to minimize the angle-of-attack variations along the blade when the flight velocity V changes. Explicit equations show how the commonly used constant-pitch ratio produces an increasing blade angle of attack as $\chi = r/r_0$ decreases, until a maximum is attained at $\chi < \frac{1}{2}$. This α max increases and moves nearer to the propeller hub when either the flight velocity is decreased or the rotational speed is increased; however, as the flight velocity approaches zero, α can become so large that it results in an unnecessarily low takeoff thrust. On the other hand, the constant α propeller has a more rapid thrust decrease as the flight velocity is increased. In any case the usual procedure is to select a blade angle β at $\chi = \frac{3}{4}$ as $\beta_{3/4} = \phi_{3/4} + \alpha_{3/4}$, where ϕ is the angle between the resultant forward velocity $(1+\delta)V$ and the rotational velocity $\frac{3}{4}r_0\omega(1-\delta')$, as shown in Fig. 1. The angle of attack (or incidence) of $\alpha_{3/4} \approx 6$ deg is selected to give the maximum lift-drag (L/D) ratio for nearly all propeller blade profiles when $\alpha_{3/4}$ measured from the profiles zero-lift reference line. For example the Clark-Y (12%) aspect ratio 6 wing (Ref. 1, p. 160) has maximum L/D at $\alpha = 5.3$ deg measured from zero lift, which corresponds to 0.3 deg relative to the flat bottom, the reference line commonly used (Fig. 1).

The angle ϕ between the plane of rotation and the resultant velocity is given at any $\chi = r/r_0$ by

$$\tan\phi_\chi = \frac{(1+\delta)V}{\chi r_0\omega(1-\delta')} = \frac{C_2(V/\omega)}{\chi} \approx \frac{(1+\delta)V}{\chi r_0\omega} \tag{1}$$

where C_2 is assumed to be a constant along the blade $(0 < \chi \leq 1)$ for a fixed flight velocity V at the given rotational speed ($\omega = 2\pi$ rpm/60), which is based on the usual assumption that the induced inflow velocity $v_i = \delta V$ is approximately uniform into the propeller disc. However in the calculations presented in Table 1 the induced rotational velocity is taken as $\delta' = 0$ because it is not as uniform, but fortunately is much smaller (Ref. 1, p. 348, $\delta' < \delta/10$). A first approximation for δ can be obtained from simple momentum theory (Ref. 2, p. 188, or Ref. 3, p. 345) where the thrust T, assumed uniform over the propeller disc area $A = \pi r_0^2$, corresponds to the inflow ratio $\delta = v_i/V$ given by

$$\delta^2 + \delta = T/4qA \qquad \text{where} \qquad q = \tfrac{1}{2}\varrho V^2$$

$$\delta = \tfrac{1}{2}\left[-1 + \sqrt{1 + (T/qA)}\right] \approx \tfrac{1}{4}(T/qA) - \tfrac{1}{16}(T/qA)^2 + \tfrac{1}{32}(T/qA)^3 + \cdots$$

Several textbooks and published reports have used the first term to approximate $\delta \approx (T/4qA)$; however, this is not satisfactory for $\delta > 0.1$. It is always better to use the exact solution because the series expansion is divergent for $(T/qA) > 1$.

Solies[4] has shown how difficult it is to calculate the propeller thrust and inflow in the general case. Fortunately, a lower limit for δ is easily obtained for light airplanes by noting that $T = C_D qS + W\sin\theta$, while $W\cos\theta = L = C_L qS \approx W$ for steady flight, either level or at the small climb angles of θ of most light airplanes. In this case a satisfactory lower limit for δ is given by

$$\delta \geq \tfrac{1}{2}\left[-1 + \sqrt{1 + \left(C_D S / \pi r_0^2\right)}\right] \tag{2}$$

More important than any variation in δ is the determination of the blade angle β that would produce a constant angle of attack α_* along most of the propeller blade at the cruising speed of a light airplane with a fixed-pitch propeller. α_* designates the airfoil profile angle of attack that produces its maximum L/D ratio. The simplest relation

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Table 1 Values of C_2 and $\phi_{3\triangleleft 4}$ for selected flight speeds V_p and r_0 = 0.914 m (31 ft)

V_p, m/s	(C_L/C_D)	δ, Eq. (2)	$C_2(V_p/\omega_p)$[a]	$\phi_{3/4}$,[b] deg	C_1[c]	(α_m/χ_m)[d]	$\phi_{3/4}$,[e] deg	(α_m/χ_m)[d]
V_{mp}(29.0)	(1.24/0.12)	0.140	0.1280	9.69	0.2106	(14.1 deg/0.16)	14.36	(10.6 deg/0.23)
V_*(38.2)	(0.714/0.06)	0.074	0.1587	11.95	0.2430	(12.1 deg/0.20)	17.61	(9.1 deg/0.28)
V_C(56.0)	(0.332/0.037)	0.046	0.2266	16.81	0.3155	(9.4 deg/0.27)	24.38	(7.4 deg/0.39)
V_M(69.5)	(0.216/0.033)	0.042	0.2802	20.49	0.3737	(8.2 deg/0.32)	29.27	(6.7 deg/0.47)

[a] $C_2(V_p/\omega_p) = (1+\delta_p)V_p/r_0\omega_p$; $\omega_p = (2700/60)2\pi$. [b] $\phi_{3/4} = \tan^{-1}(C_2/0.75)$. [c] Equation (3), $\beta_{3/4} = \phi_{3/4} + 6$ deg. [d] Equation (5).
[e] $\tan^{-1}(1.5C_2/0.75)$; $\omega_p = (1800/60)2\pi$.

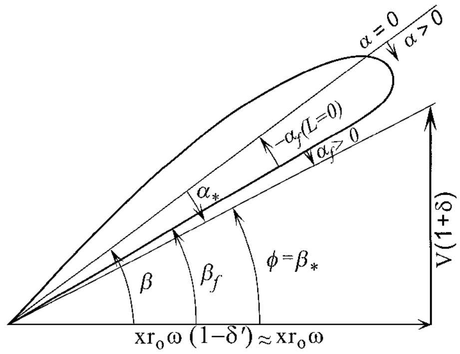

Fig. 1 Tan $\phi \approx (1+\delta)V/\chi r_0\omega = C_2/\chi$. When $\beta_* = \phi \approx \tan^{-1}(C_2/\chi)$, then $\alpha = \alpha_*$. When $-\alpha_f(L=0) = \alpha_*$, then $\beta_* = \beta_f = \phi$, and $\alpha_f = 0$.

for β is given by the constant-pitch ratio defined by the pitch p and the blade-tip radius r_0 as

$$p/2\pi r_0 = \chi \tan\beta_\chi = \text{const} = C_1$$

where $\chi = r/r_0 \leq 1$. However, when β is measured from the zero-lift reference line (Fig. 1), then $\beta_\chi = (\phi_\chi + \alpha_*) = \tan^{-1}(C_1/\chi)$. Because both β and ϕ vary similarly with χ, α_* is approximately constant (5 deg < α_* < 6 deg) for most useful airfoils; therefore, as will be shown by Eqs. (3) and (4), $\alpha_\chi = (\beta_\chi - \phi_\chi)$ increases from the propeller tip ($\chi = 1$) to a maximum α_m at χ_m near the propeller hub. To reduce this increase in α_χ as χ decreases, the highly cambered flat bottom Clark-Y and RAF-6 profiles were developed so that (see Fig. 1) $|\alpha_f(L=0)| = 5$ deg $\approx \alpha_*$. Then when $\alpha_f = 0$ and β_f measured from the flat lower surface,

$$\beta_f = \phi, \qquad \beta = (\beta_f + 5\,\text{deg})$$

$$\alpha = (\beta - \phi) = |\alpha_f(L=0)| = 5\,\text{deg} \approx \alpha_*$$

$$p/2\pi r_0 = \chi \tan\beta_{f,\chi} = \chi \tan\phi_\chi = C_2$$

This could be the ideal solution if all of the propeller sections had a 12% thick Clark-Y profile having $|\alpha_f(L=0)| = 5$ deg $\approx \alpha_* =$ 5.3 deg; unfortunately structural demands require that blade thickness decreases from hub to tip. Although α_* is relatively unaffected by thickness, (5 deg $\leq \alpha_* <$ 6 deg) for most useful profiles, $|\alpha_f(L=0)|$ decreases with thickness ratio $|\alpha_f(L=0)| \approx 44(t/c)$. For example, a 6% Clark-Y airfoil has $|\alpha_f(L=0)| \approx$ 3 deg < $\alpha_* \approx$ 5 deg. Therefore, the present discussion indicates how the pitch ratio must be modified to reduce this increase in α_χ as χ decreases.

II. Fixed-Pitch Propeller with Constant Pitch

A constant geometric pitch p is defined for $\chi = r/r_0$,

$$p/2\pi r_0 = (\chi \tan\beta_\chi) = C_1 = \left(\tfrac{3}{4}\tan\beta_{\frac{3}{4}}\right) \tag{3}$$

The usual notation rated pitch is for $\chi = \frac{3}{4}$ so that $p = 2\pi r_0(\frac{3}{4}\tan\beta_{3/4})$ is marked on the propeller hub, along with the propeller diameter ($2r_0$). When the blade angles β are measured from the zero-lift reference line (Fig. 1, $\alpha = 0$) and Eq. (3) is defined by $\beta_{3/4} = \phi_{3/4} + \alpha_{3/4}$, then the blade angles of attack α_χ will not be constant along the blade, as shown by von Mises[1] (p. 292). The variation of the angle of attack along the blade $0.15 < \chi = r/r_0 \leq 1$ is given by (see Fig. 1 and dotted line in Fig. 2)

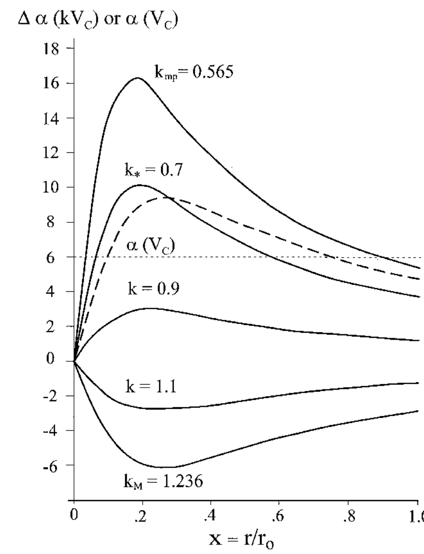

Fig. 2 Variation of $\Delta\alpha(kV_c)$ with $\chi = r/r_0$. From Eqs. (8) and (10) and $\alpha(V_c)$ from Eq. (4) with V_c = 125 mph (59 m/s), $C_2(V_c)$ = 0.2266, and C_1 = 0.3155 for $\beta_{3\triangleleft 4} = \phi_{3\triangleleft 4}$ + 6 deg.

$$\alpha_\chi = \beta_\chi - \phi_\chi = \tan^{-1}\frac{C_1}{\chi} - \tan^{-1}\frac{C_2}{\chi} = \tan^{-1}\left(\frac{\chi(C_1 - C_2)}{\chi^2 + C_1C_2}\right) \tag{4}$$

Consequently α_χ increases as χ decreases to $\chi_m \leq \chi \leq 1$ so that the maximum α_m is

$$\alpha_m = \tan^{-1}[(C_1 - C_2)/2\chi_m], \qquad \chi_m = (C_1C_2)^{\frac{1}{2}} \tag{5}$$

For $\chi < \chi_m$, α_χ decreases to zero because $\tan\alpha_\chi \to \chi(1/C_2 - 1/C_1) \to 0$ as $\chi \to 0$.

The numerical values in Table 1 were calculated for a typical light airplane, similar to a Piper PA-22 or a Cessna 170. The drag polar approximation was given by

$$C_D = C_{D_e} + C_L^2\big|\ \pi Ae = 0.03 + C_L^2\big|\ 17$$

$$C_{L*} = \left(\pi AeC_{D_e}\right)^{\frac{1}{2}} = 0.714, \qquad C_{D_*} = 2C_{D_e} = 0.06$$

$$C_{L_{mp}} = 3^{\frac{1}{2}}C_{L*} = 1.24, \qquad C_{D_{mp}} = 2C_{D_*} = 0.12$$

The velocity V_* for the maximum L/D ratio, and V_{mp} for the minimum power required for steady level flight, were then calculated by

$$V = (2W/\rho SC_L)^{\frac{1}{2}}, \qquad W = 907\,\text{kg}\,(2000\,\text{lb})$$

$$S = 13.9\,\text{m}^2\,(150\,\text{ft}^2), \qquad \rho = 1.225\,\text{kg/m}^3\,(0.002377\,\text{lb s}^2/\text{ft}^4)$$

The constant $C_2 = (1+\delta)V/(r_0\omega)$ was calculated for $r_0 = 0.914$ m (3 ft) and $\omega = 90\pi$ (2700 rpm). For $\omega = 60\pi$ (1800 rpm) the constant

C_2 in Table 1 was simply increased by a factor of 1.5. The constant $C_1 = \frac{3}{4}\tan(\phi_{3/4} + 6\text{ deg})$ in Table 1 was calculated for $\alpha_{3/4} = 6$ deg, measured from the zero-lift reference line to be compared with the calculations given by von Mises.[1]

The cruise speed V_C in Table 1 was selected as the advertised 125 mph (56 m/s) cruise speed of the Piper PA-22. The maximum speed V_M could only be attained with a variable pitch propeller, but is given to show how the maximum α_m decreases with increasing speed. At lower speeds the increase in angle of attack as χ decreases can become too large when $\alpha_{3/4} = 6$ deg is measured from the zero-lift reference line. For example, at $V_{\text{mp}} = 29$ m/s, $\alpha_m = 14.1$ deg at $\chi_m = 0.164$, whereas at V_C (56 m/s), α_m has decreased to 9.43 deg and moved outward to $\chi_m = 0.267$ so that the blade angle of attack has decreased to 8.07 deg at the propeller hub ($\chi = 0.15$).

The excessive increase in α can be easily avoided by measuring β from a reference line below that for zero lift, as indicated in Fig. 1. For example, the 12% Clark-Y has a flat lower surface so blade angles ($\beta_{f,\chi}$) measured from its bottom surface can be defined as $\beta_{f,\chi} = (\beta_\chi - 5\text{ deg})$ (see Ref. 1, p. 160), while $\alpha_* \approx 5.3$ deg; therefore, $\beta_{f,3/4} = (\phi_{3/4} + 0.3\text{ deg})$ would correspond to the blade profile's maximum ratio of L/D. The best procedure is to neglect the 0.3 deg and use $\beta_{f,\chi} = \phi_\chi$ and $C_1 = \frac{3}{4}\tan(\phi_{3/4}) = \chi\tan\phi_\chi$ for a redefined constant pitch given by

$$p = 2\pi r_0(\chi\tan\phi_\chi) = 2\pi r_0 C_2(V_p/\omega_p) = 2\pi r_0\chi\,(\tan\beta_{f,x})$$

Consequently for the Clark-Y profile shown in von Mises[1] (p. 160), $\alpha_\chi = (\alpha_{f,\chi} + 5\text{ deg})$, and $C_{L*}/C_{D*} \approx 22$ for $-2\text{ deg} < \alpha_{f,\chi} < 3$ deg or ($3\text{ deg} < \alpha_\chi < 8$ deg). The aerodynamic angle of attack (5 deg) is constant along the blade portion that has the Clark-Y profile at the design advance ratio given by $C_2(V_p/\omega_p) = p/2\pi r_0$, when $\alpha_f = 0$. Unfortunately the 5-deg zero-lift angle decreases as the 12% thickness decreases, becoming only 3 deg for 6% thickness so that α would decrease at the thinner outboard sections unless $\beta_{f,\chi}$ was increased by $\alpha_{f,\chi}$ (see Fig. 1), where $0 \le \alpha_{f,\chi} \le 2$ deg.

III. Fixed-Pitch Propeller with Constant α

As noted by von Mises,[1] a fixed-pitch propeller that has β_χ for the constant pitch defined by Eq. (3) with $\beta_\chi > \phi_\chi$ cannot have a constant angle of attack. This is clearly shown by Eq. (4) because α varies with $\chi = r/r_0$ whenever $C_1 > C_2$. However, when $C_1(V_p/\omega_p) = C_2(V_0/\omega_0)$, then $\alpha_\chi = 0$, and $(p/2\pi r_0) = (V_0/r_0\omega_0)$, with $\delta = 0 = \alpha$ when β_χ and α_* are measured from the zero-lift reference line. The fixed-blade angles necessary for a constant $\alpha_\chi = \alpha_*$ (the maximum L/D ratio for the blade profile) are given by $C_* = \tan\alpha_*$ so that

$$\beta_\chi = \alpha_* + \phi_\chi = \tan^{-1}C_* + \tan^{-1}(C_2/\chi) = \tan^{-1}\left(\frac{C_* + C_2/\chi}{1 - C_*C_2/\chi}\right)$$

$$C_*C_2 < \chi = (r/r_0) \le 1$$

$$(p_\chi/2\pi r_0) = \chi\tan\beta_\chi = \chi\left(\frac{C_* + C_2/\chi}{1 - C_*C_2/\chi}\right) \tag{6}$$

Consequently the varying pitch (p_χ) decreases with χ until a minimum is attained at χ_p, which is evaluated by $(dp_\chi/d\chi) = 0$ as

$$\chi_p = C_*C_2\left[1 + \left(1 + C_*^{-2}\right)^{\frac{1}{2}}\right] = 1.11C_2 \quad \text{for} \quad (C_* = \tan 6\text{ deg})$$

This variation of p_χ is confirmed by von Mises[1] (Fig. 220, p. 292) if J/π is replaced by C_2 with $\delta = 0$, which is adequate for higher velocities and lower C_D than those presented in Table 1. Note that the δ values given by Eq. (2) are only lower limit estimates.

The variation in p_χ decreases as the reference line for β is moved downward below the zero-lift line. A constant pitch $p = 2\pi r_0 C_2$ is attained when the blade pitch angle $\beta_{*,\chi} = \phi_\chi$, as shown in Fig. 1. Then with $\alpha_\chi = \alpha_* =$ constant,

$$\beta_{*,\chi} = \beta_\chi - \alpha_* = \phi_\chi = \tan^{-1}(C_2/\chi)$$

$$\beta_\chi = \beta_{*,\chi} + \alpha_* = \alpha_* + \phi_\chi \tag{7}$$

This agrees with Eq. (6), except that the measured pitch is now constant because $p/2\pi r_0 = \chi\tan\phi_\chi = C_2$. The constant $\alpha_\chi = \alpha_*$ could be produced from the propeller hub ($\chi = 0.15$) to its tip ($\chi = 1$) as long as the advance ratio $C_2(V_p/\omega_p) =$ constant. Any change in V or ω will vary α_χ along with the blade, as shown by

$$\alpha_\chi(V/\omega) - \alpha_* = \phi_\chi(V_p/\omega_p) - \phi_\chi(V/\omega) = \tan^{-1}\frac{C_2(V_p/\omega_p)}{\chi} - \tan^{-1}\frac{C_2(V/\omega)}{\chi} = \tan^{-1}\left\{\frac{\chi[C_2(V_p/\omega_p) - C_2(V/\omega)]}{\chi^2 + C_2(V_p/\omega_p)C_2(V/\omega)}\right\} = \Delta\alpha_\chi \tag{8}$$

Obviously, α_χ decreases as either V increases or ω decreases and remains constant (α_*) only if the advance ratio, defined by $C_2(V_p/\omega_p)$, remains constant.

The maximum change in $[\alpha_\chi(V/\omega) - \alpha_*] = \Delta\alpha_\chi$, given by $(d\Delta\alpha/d\chi) = 0$, is

$$\tan^{-1}\left(\frac{C_2(V_p/\omega_p) - C_2(V/\omega)}{2\chi_m}\right)$$

$$\chi_m = [C_2(V_p/\omega_p)C_2(V/\omega)]^{\frac{1}{2}} \tag{9}$$

Substituting $k = C_2(V/\omega)/C_2(V_p/\omega_p)$ into Eq. (9) simplifies it to

$$\Delta\alpha_m(kV_p) = \tan^{-1}\left[(1-k)/2\sqrt{k}\right], \qquad \chi_m = \sqrt{k}C_2(V_p/\omega_p) \tag{10}$$

The values for $\Delta\alpha(kV_C)$ shown in Fig. 2 were calculated by Eqs. (8) and (10) for velocity changes from an initial advance ratio given in Table 1 by $C_2(V_C/\omega_C) = 0.2266$, corresponding to $V_C = 56$ m/s and $\omega_C = 90\pi$(2700 rpm) so that $k = (1+\delta)V/58.6$ m/s. The effect of any changes in rpm can be calculated by simply changing the value of C_2 by the ratio (2700/rpm). Figure 2 shows how quickly the blade angles of attack can be increased by a decrease in velocity. For example, the decrease in velocity from $V_C = 56$ m/s to $V_{\text{mp}} = 29$ m/s ($k_{\text{mp}} = 0.565$) produces $\Delta\alpha_{0.4} = 12$ deg so that $\alpha_{0.4}(V_C)$ has increased from $\alpha_* = 6$ deg to $\alpha_{0.4}(V_{\text{mp}}) = 18$ deg. This would correspond to a decrease in the L/D ratio of a typical Clark-Y profile by more than 50%. On the other hand, an increase in velocity to $V_m = 69.5$ m/s ($k_m = 1.24$) produces $\alpha_\chi(V_m) \approx 0$ for $0.2 \le \chi \le 0.35$, when the initial $\alpha_\chi(V_C) = 6$ deg. This shows why a variable-pitch propeller is necessary for both a desirable speed range and a reasonable takeoff distance.

However the best compromise for a constant-pitch propeller for a light airplane has a blade profile that has a flat lower surface and a large camber. The flat bottom is suitable for verifying the blade-pitch angles with a propeller-protractor, and the high camber can lead to $-\alpha_f(L = 0) \approx \alpha_* \approx 5$ deg, where the angle α_f is measured from the flat bottom [see Fig. 1 ($\alpha = \alpha_f + 5$ deg)]. Both the Clark-Y and the RAF-6 profiles are near to this ideal condition. For example, a 12% Clark-Y profile (Ref. 1, p. 160) has $\alpha = 5\text{ deg} \approx \alpha_* = 5.3$ deg when $\alpha_f = 0$. Consequently the optimum $\alpha_* \approx 5$ deg is closely approximated by

$$\beta_\chi = \phi_\chi + 5\text{ deg} = \beta_{f,\chi} + 5\text{ deg}, \qquad \alpha_\chi = \alpha_{f,\chi} + 5\text{ deg}$$

$$\beta_{f,\chi} = \phi_\chi = \tan^{-1}(C_2/\chi), \qquad p/2\pi r_0 = C_2 = \chi\tan\phi_\chi \tag{11}$$

This is the ideal solution if the blade thickness remained 12%; however, structural requirements need a decrease in thickness from hub to tip. Consequently the outboard profiles must increase $\beta_{f,\chi}$ to compensate for the decrease in $-\alpha_f(L = 0)$.

IV. Discussion

The $\Delta\alpha$ from Eqs. (8) and (10) and Fig. 2 can also be applied to the constant-pitch propeller defined by Eq. (3) with $\beta_{3/4} = \phi_{3/4} + \alpha_{3/4}$ because

$$\Delta\alpha_\chi(kV_C) = \phi_\chi(V_C) - \phi_\chi(kV_C)$$

$$\alpha_\chi(kV_C) = \beta_\chi(V_C) - \phi_\chi(kV_C) = \alpha_\chi(V_C) + \Delta\alpha_\chi(kV_C)$$

The variation of $\alpha_\chi(V_C) = \beta_\chi(V_C) - \phi_\chi(V_C)$ is shown by the dotted line in Fig. 2 for $C_1(V_C) = 0.3155$ from Table 1 for $\alpha_{3/4} = 6$ deg at 2700 rpm. Now the angle of attack $\alpha_\chi(kV_C)$ is increasingly greater for $\chi < \frac{3}{4}$ as the velocity decreases; however, an increase in velocity $(k > 1)$ decreases $\alpha_\chi(kV_C)$ so as to make it more nearly constant. Unfortunately it does not become constant until $\alpha_\chi(k_0V_C) = 0$, when

$$\beta_\chi(V_C) = \phi_\chi(k_0V_C), \qquad C_1(V_C) = C_2(k_0V_C) = k_0V_c/r_0\omega_0$$

because $\alpha = 0 = \delta_0$. Equation (9) gives the maximum value of $\alpha_\chi(kV_C)$ only when $\alpha_\chi(V_C)$ is constant. When it varies, then $\alpha_m(kV_C)$ is given by Eq. (5) at a larger value of χ_m.

So far all calculations (except those in Table 1 for $\omega = 60\pi = 1800$ rpm) have been based on 2700 rpm. Now it will be shown how a variation in rpm while changing velocity can extend the range of constant angle of attack for any fixed-pitch propeller. For example, the increase in $\Delta\alpha$ shown in Fig. 2 for $k = 0.9$ can be eliminated by decreasing the rpm from 2700 to 2430 rpm because $\delta < 0.05$ can be considered constant:

$$C_2(V_C/\omega_C) = C_2(kV_C/k\omega_C) = 0.2266, \qquad \Delta\alpha_\chi = 0$$

However the decrease in $\Delta\alpha$ for $k = 1.1$ cannot be eliminated by increasing the rpm, which must be limited to 2700 rpm for continuous operation so that the pitch ratio for the constant $\alpha(V_C) = \alpha_*$ propeller, $p/2\pi r_0 = C_2(V_C/\omega_C) = 0.2266$, must be increased. To assist the average light plane owner, select the appropriate propeller pitch and consider the application to a desired cruise speed $V_C = 125$ mph to be compatible with $V_m = 135$ mph. Then $V_m/V_C = 1.08 = 2700$ rpm/2500 rpm, and $\alpha(V_C) = \alpha(1.08V_C) = \alpha_*$ for maximum blade L/D ratio if $\omega_C = (90\pi/1.08) = 2500$ rpm, so that

$$C_2(125/2500) = C_2(135/2700) = 1.08(0.2266) = 0.2447$$

$$p = 2\pi r_0 C_2(125/2500) = 55.36\,\text{in.} = 72\,\text{in.}\,\pi(0.2447)$$

$$\beta_{f,\frac{3}{4}} = \phi_{\frac{3}{4}} = \tan^{-1}(0.2447/0.75) = 18.07\,\text{deg}$$

From Table 1 note that originally $\phi_{3/4} = 16.81$ deg for $C_2(V_C/\omega_C) = 0.2266 = C_2(125$ mph/2700 rpm) so that originally $p = 72\,\text{in.}\,\pi(0.2266) = 51.26\,\text{in.} = (55.36\,\text{in.}/1.08)$. The change in δ is negligible for this relatively small velocity increase from 125 mph (56 m/s). The older U.S. light airplanes in operation today record their air speed in miles per hour, but the more recent ones are operating in knots. Table 1 is based on meters per second (m/s) as given by

$$(\text{m/s}) = (\text{ft/s}) \times 0.3048 = (\text{mph}) \times 0.44704 = (\text{kn}) \times 0.51444$$

$$= (\text{km/hr}) \div 3.6$$

The lower limit for δ, as given by Eq. (2), is $\delta < 0.1$ for $(C_DS/\pi r_0^2) < 0.44$. For Table 1 this corresponds to $V > 33$ m/s (73.8 mph). However for lower velocities the estimation of δ becomes difficult, as shown by Solies[4] who calculated $\delta = 0.57$ at 50 kn (25.7 m/s) for a propeller with 2.08 m (6.83-ft) diameter developing 508 kg (1120 lb) of thrust at 2700 rpm. Consequently the estimates for δ in Table 1 when $V < 33$ m/s could be in the range given by

$$(T/4qA) - (T/4qA)^2 < \delta < \tfrac{1}{2}$$

The δ values in Table 1 were calculated by Eq. (2), which is not applicable for lower velocities because $(T/qA) > (C_DS/qA)$.

To estimate the range of the blade angle $\beta_{f,3/4}$ that would correspond to the estimated range of δ, consider the lowest velocity in Table 1:

$$V_{\text{mp}} = 29\ \text{m/s}, \qquad \delta = 0.14\ [\text{from Eq. (2)} \quad \text{with} \quad C_D = 0.12]$$

$$C_2 = 0.128, \qquad \phi_{\frac{3}{4}} = \tan^{-1}(0.128/0.75) = 9.7\,\text{deg}$$

The method of Solies[4] gives $\delta = 0.37$ (a 264% increase) so that $1.14 < (1 + \delta) \le (1.37)$ and $k \le (1.37/1.14) = 1.20$. Then with the Clark-Y profile (Ref. 1, p. 160), $|\alpha_f(L = 0)| = 5\,\text{deg} \approx \alpha_*$, and $\alpha_\chi(kC_2) = 5$ deg when $\alpha_f = 0$ and $\phi(kC_2) = \beta_f$ in Fig. 1. If $\delta = 0.14$, then $\alpha_\chi(0.128) = 5$ deg when $\beta_{f,3/4} = \phi_{3/4}(0.128) = 9.7$ deg; however, if $\delta = 0.37$ and $k = 1.2$, then $\alpha_\chi(0.154) = 5$ deg if $\beta_{f,3/4} = \phi_{3/4}(0.154) = \tan^{-1}(0.154/0.75) = 11.6$ deg. In this case the fixed-pitch propeller for climb at $V_{\text{mp}} = 29$ m/s can be defined by

$$0.128 < (C_2 = p/2\pi r_0) < 1.2(0.128) = 0.154$$

$$9.7\,\text{deg} < \left(\beta_{f,\frac{3}{4}} = \phi_{\frac{3}{4}}\right) < 11.6\,\text{deg}$$

In many cases the actual profile used has

$$|\alpha_f(L = 0)| \approx 4\,\text{deg}, \qquad \alpha_* < 6\,\text{deg}$$

So the usual procedure is to define the blade angle by

$$\beta_{f,\frac{3}{4}} = \phi_{\frac{3}{4}}(C_2) + \alpha_{f,\frac{3}{4}} \qquad \text{with} \qquad \alpha_{f,\frac{3}{4}} \approx 2\,\text{deg}$$

In this case $\alpha_\chi(C_2)$ cannot be constant if the pitch is defined by Eq. (3), as shown by Eqs. (4) and (6), because α_χ can be constant only when $\phi(C_2)$ coincides with the reference line used to define the blade angle (e.g., Fig. 1, $\phi = \beta_*$ for $\alpha_* = $const). When β_f is measured from the flat lower surface ($\alpha_f = 0$), then $\alpha_\chi(kC_2)$ is constant for $kC_2 > C_2$ so that

$$\phi_{\frac{3}{4}}(kC_2) = \beta_{f,\frac{3}{4}} = \phi_{\frac{3}{4}}(C_2) + 2\,\text{deg}$$

$$k = \tan\phi(kC_2)/\tan\phi(C_2)$$

$$\alpha(kC_2) = \beta_{f,\chi} + |\alpha_f(L = 0)| - \phi_\chi(kC_2) = |\alpha_f(L = 0)| \tag{12}$$

The variation of $\alpha(C_2)$ is given by Eq. (8) as

$$-\Delta\alpha_\chi(kC_2) = \phi_\chi(kC_2) - \phi_\chi(C_2) = \tan^{-1}\left[\frac{\chi(k-1)C_2}{\chi^2 + kC_2^2}\right] \tag{13}$$

For $k < 1$ $\alpha_\chi(C_2)$ increases as χ decreases to χ_m defined by Eq. (10) for maximum α_m.

The preceding calculations for $V_{\text{mp}} = 29$ m/s gave $C_2 = 0.128$ and $\phi_{3/4}(0.128) = 9.7$ deg. Then $\phi_{3/4}(kC_2) = (9.7\,\text{deg} + 2\,\text{deg}) = \beta_{f,3/4}$ so that $k = (\tan 11.7\,\text{deg}/\tan 9.7\,\text{deg}) = 1.212$; consequently if $|\alpha_f(L = 0)| = 4$ deg, then Eq. (13) gives the following $\alpha_\chi(0.128)$ values:

$$\alpha_1 = 5.5\,\text{deg}, \qquad \alpha_{\frac{3}{4}} = 6\,\text{deg} = \alpha_*$$

$$\alpha_m = 9.5\,\text{deg}(\chi_m = 0.141)$$

Although $\alpha_\chi(0.128)$ is not constant, it may be useful because it becomes more uniform as C_2 increases to $kC_2 = 1.212(0.128)$ when $\alpha_\chi(0.155) = |\alpha_f(L = 0)| = 4$ deg. This constant α_χ occurs at a velocity that is more than 21% greater than the original velocity (29 m/s), as given by

$$V = 1.212[1.14/(1 + \delta)](\text{rpm}/2700)(29) > 35\,\text{m/s}$$

because δ decreases and rpm increases, as V increases.

The δ values used have a large effect on β, but a very small effect on the variation of $\alpha_\chi(C_2)$, as can be shown by replacing the lower limit $\delta = 0.14$ and $C_2 = 0.128$ by the upper limit $\delta = 0.37$ and $C_2 = 0.154$, corresponding to $\phi_{3/4}(0.154) = 11.6$ deg so that for this application $\beta_{f,3/4} = \phi_{3/4} + 2\,\text{deg} = 13.6$ deg and $k = (\tan 13.6\,\text{deg})/(\tan 11.6\,\text{deg}) = 1.179$. This value of k less than

3% lower than $k = 1.212$ for $\delta = 0.14$. As expected, Eq. (13) gives $\alpha_{3/4} = 6$ deg in both cases, but $k = 1.179(C_2 = 0.154)$ gives $\alpha_m = 8.7\,\mathrm{deg}(\chi_m = 0.167)$, which is only 0.8 deg less than $\alpha_m = 9.5\,\mathrm{deg}(\chi_m = 0.141)$ for $\delta = 0.14$. Consequently, the main effect of $0.14 < \delta < 0.37$ is the 1.9 deg increase in $\beta_{f,3/4}$ from 9.7 to 11.6 deg in order to match the increase in the inflow velocity (δV) at $V = 29$ m/s, with constant $\alpha_\chi = 4$ deg occuring at $V \approx 354$ m/s. The best adjustment for the estimated variations in either δ or $|\alpha_f(L = 0)|$ is to use $1.2C_2$ rather than $\beta_{f,3/4} = \phi_{3/4} + 2$ deg for this particular case. Although α_* can increase to 6 deg for other profiles, $|\alpha_f(L = 0)|$ can be reduced to 3 deg by thinner Clark-Y profiles having approximately 6% thickness, as indicated by McCormick[3] (p. 356). In this case α_χ is not constant if the constant pitch is based on $\beta_f = \phi$, because $\alpha_{3/4} = \beta_{f,3/4} + 5\,\mathrm{deg} - \phi_{3/4}$ and at the propeller tip ($\chi = 1$), $\alpha_1 = \beta_{f,1} + 3\,\mathrm{deg} - \phi_1$. Consequently the outboard profiles must have their blade angles $\beta_{f,\chi}$ increased with respect to ϕ_χ so $\beta_{f,\chi} = \phi_\chi + \alpha_{f,\chi}$, as shown in Fig. 1, with $\alpha_{f,1} = 2$ deg, to have $\alpha_\chi \approx 5$ deg. To accommodate the thinner profiles, most variable-pitch propellers refer to the blade angles as β from zero lift (see Fig. 1) so that as the blades are rotated to increase β for faster flight speeds, the resulting α_χ are approximately constant (von Mises,[1] Fig. 221, p. 293).

A constant $\alpha_\chi = \alpha_*$ for the maximum L/D ratio is an obvious goal because it maximizes the thrust-to-torque ratio. Unfortunately the bending and tension stresses produced in flight require such a thickening of the inboard blade profiles that their L/D ratios are greatly reduced. In addition, during the takeoff run ϕ is so small that $\alpha \approx \beta = \beta_f + |\alpha_f(L = 0)|$ produces blade stall with further reduction in the L/D ratio for $\chi < \frac{1}{2}$. Consequently most commercially produced fixed-pitch propellers gradually reduce the constant pitch for $\chi < 0.7$. Some variable-pitch propellers have constant pitch for $\chi > 0.35$. For example, McCormick[3] (Fig. 6.11, p. 355) shows pitch p variation with χ for a 10-ft diameter, three-blade propeller with Clark-Y profiles, and $(p/2\pi r_0) = \frac{3}{4}\tan 15\,\mathrm{deg} = 0.201$ for $0.35 < \chi \leq 1$. This is reduced 36% to $(p/2\pi r_0) = 0.148$ at $\chi = 0.2$, where $\beta_{0.2} = \tan^{-1}(0.148/0.2) = 36.5$ deg, which is 8.6 deg less than $\beta_{0.2} = \tan^{-1}(0.201/0.2) = 45.1$ deg for the original constant pitch. The majority of the replacement propellers for U.S. light airplanes are manufactured by the McCauley Industrial Corp., Vandalia, Ohio. They provide fixed-pitch metal propellers that have their diameter ($2r_0$) and pitch ($p = 2\pi r_0 \times 0.75 \times \tan\beta_{f,3/4}$) stamped on their hub in inches (0.0254 m = 1 in.). Suitable for the light airplane defined by Table 1, they produce 72-in. diameter propellers with fixed-pitch ranging from 46 to 58 in. at 2-in. intervals. Each 2-in. increase in pitch increases $\Delta\beta_{f,3/4} \approx 0.6$ deg. This corresponds to an increase of $\Delta V_p \approx 5$ mph. The design V_p for the best cruise speeds could then be approximately 110 mph for $p = 46$ in., 125 mph for $p = 52$ in., or 140 mph for $p = 58$ in. These McCauley propellers have slightly increasing pitch for $\chi > \frac{3}{4}$, and then a rapidly decreasing pitch toward the hub so that at $\chi = \frac{1}{3}$ the blade angles β_f (see Fig. 1) are 5–7 deg less than the corresponding values given by $\chi\tan\beta_{f,\chi} = \frac{3}{4}\tan\beta_{f,3/4}$. Then near the hub ($\chi \leq \frac{1}{4}$)$\beta_f$ is kept constant, whereas the calculated values given by $\beta_{f,\chi} = \tan^{-1}(p_{3/4}\mathrm{in.}/2\pi r\ \mathrm{in.})$ rapidly increase because $\beta_f \to 90$ deg as r in. $\to 0$. This decrease in $\beta_{f,\chi}$ for $\chi < \frac{3}{4}$ was developed by McCauley's engineers to increase the static thrust and alleviate the high angle-of-attack airfoil profile stall that occurs as V decreases. As shown by Eq. (8) and Fig. 2, any decrease in velocity produces an undesirable increase in α_χ for the inboard blade profiles ($0.2 < \chi < 0.6$).

If the propeller selection is based on a design $V_p \geq 125$ mph, then the inflow velocity can be neglected ($\delta < 0.046$). However δ should be considered for the low-speed ultralight airplane, and especially for the man-powered aircraft. For the latter case it is better to evaluate both the axial and radial inflow and evaluate β_χ from the zero-lift reference. For a first approximation Eq. (6) would indicate how the pitch would vary to produce $\alpha_\chi \approx \alpha_*$. However these calculations would be too expensive for the home-built airplane designer so that their simplest procedure would be to estimate the axial inflow by $(C_D S/4\pi r_0^2) < \delta < \frac{1}{2}$, and then use the constant pitch defined by

$$p/2\pi r_0 = \tfrac{3}{4}\tan\beta_{f,\frac{3}{4}} = \chi\tan\beta_{f,\chi}, \qquad \beta_{f,\frac{3}{4}} = \phi_{\frac{3}{4}} + \Delta\alpha_{f,\frac{3}{4}} \quad (14)$$

where β_f is measured from the flat lower surface of a typical Clark-Y airfoil. For example, $\Delta\alpha_{f,3/4} = 2$ deg would produce $\alpha_\chi \approx \alpha_*$ for $\chi > 0.6$ if the Clark-Y thickness ratio was 10% at $\chi = \frac{3}{4}$ and 6% at $\chi = 1$. In addition $\Delta\alpha_{f,3/4} = 2$ deg would reduce the undesirable increase in α_χ, which is shown in Fig. 2 for $\beta_{3/4} = \phi_{3/4} + 6$ deg. The amateur propeller builders should refrain from reducing the pitch for $\chi < 0.6$ because they would lose the simplicity of graphically determining the blade angles by drawing the straight line given by $\tan\beta_{f,\chi} = (p/2\pi r_0)/\chi$. The McCauley fixed-pitch propellers decreased the pitch following many tests to increase the thrust as the velocity decreases. As a result, they provide a table with each propeller that gives the blade angle ($\beta_{f,\chi}$) every 3 in. from hub to propeller tip (r_0 in.). Another objective in the McCauley propeller tests was to increase the static thrust, which is important for the takeoff run. If the propeller pitch is selected too large in the hope of achieving a higher $V_{\max}$, then α_χ during the takeoff run may be so large that the propellers L/D ratio is too small to achieve a relatively finite takeoff distance!

V. Conclusions

The design goal for either a fixed-pitch or a controllable-pitch propeller is to have most of the blade for $\chi > \frac{1}{4}$ at each profiles angle of attack $\alpha = \alpha_*$ for its maximum L/D ratio. This could be achieved for the design advance ratio $C_2(V_p/\omega_p)$ by using Eq. (6) to determine the blade angles β_χ (measured from the zero lift line). The required pitch (p_χ) decreases from $\chi = 1$ to $\chi_p \approx 1.11C_2\ (V_p/\omega_p)$, and then increases rapidly; therefore, $p_\chi = p_p$ for $\chi < \chi_p$ is the best choice since the required blade thickness had greatly reduced the L/D ratio. Then the decrease in α_χ that would decrease thrust as V increases, and the increase in α_x that could decrease the L/D ratio of the profiles when $\alpha_\chi \to \beta_\chi$ as V decreases, must be checked for the expected V_{MAX} and the thrust required for the take-off run. The change in α_χ produced by any change in V or ω, for any fixed β_χ and its corresponding α_χ at the design $C_2(V_p/\omega_p)$, is given by Eqs. (10) and (13), for $C_2(kV_p/\omega_p)$, where $k = (V\omega_p/V_p\omega)[(1+\delta)/(1+\delta_p)]$; the change in δ can usually be neglected if $V > 40$ m/s.

However, the best propeller for light airplanes uses airfoil profiles that have a flat lower surface, such as the Clark-Y airfoil. The flat bottom provides an excellent surface for a propeller protractor to check the blade angle β_f, thereby eliminating the difficulty of measuring β from the zero lift reference line (Fig. 1). In addition, the 12% Clark-Y profile's camber was designed so that $\alpha_f = 0$ in Fig. 1, gave

$$-\alpha_f(L = 0) = 5_{\mathrm{deg}} \approx \alpha_*, \qquad \beta_f = \phi$$

$$\alpha = \alpha_f + |(L = 0)| = 5_{\mathrm{deg}} \quad (15)$$

Now $\beta_{f,\chi}$ is defined by Eq. (11) as $\tan\beta_{f,\chi} = (C_2/\chi)$, consequently $\beta_{f,\chi}$ is easily calculated, or measured by the angle formed by a straight line graph given by r/r_0 vs $C_2 =$ constant. For a low-speed ultralight airplane this would be the ideal solution, with $(t/c) = 0.12$ for $\chi > \frac{1}{4}$; however, structural demands for heavier aircraft require a continual decrease in (t/c) for $\chi > \frac{1}{4}$. Because $|\alpha_f(L = 0)| \approx 44(t/c)$ decreases, while $\alpha_* \geq 5_{\mathrm{deg}}$ is approximately constant, the usual remedy is to increase $\beta_{f,3/4}$, as in Eq. (14) with $\Delta\alpha_{f,3/4} = (\beta_{f,3/4} - \phi_{3/4}) = \alpha_{f,3/4}$ (in Fig. 1).

For example, Table 1 gives $C_2(V_c/\omega_c) = 0.2266$ and $\phi_{3/4} = 16.81_{\mathrm{deg}}$ when $V_c = 56$ m/s (125 mph). Then $\Delta\alpha_{f,3/4} = 1.6_{\mathrm{deg}}$ and Eq. (14) give,

$$\chi\tan\beta_{f,\chi} = \tfrac{3}{4}\tan\beta_{f,\frac{3}{4}} = \tfrac{3}{4}\tan(16.81 + 1.6) = 0.2496$$

$$\alpha_{f,\chi} = \beta_{f,\chi} - \phi_\chi = \tan^{-1}(0.2496/\chi) - \tan^{-1}(0.2266/\chi)$$

$$\alpha_{f,1} = 1.25_{\mathrm{deg}}, \qquad \alpha_{f,\frac{3}{4}} = 1.6_{\mathrm{deg}}$$

$$\alpha_{f,\frac{1}{2}} = 2.15_{\mathrm{deg}}, \qquad \alpha_{f,\frac{1}{4}} = 2.77_{\mathrm{deg}}$$

Consequently for $\chi, > \frac{1}{4}$, $-\alpha_f(L = 0) \approx 3_{\mathrm{deg}}$ would produce $\alpha_\chi \approx 5_{\mathrm{deg}} \approx \alpha_*$. However a much simpler method is indicated by Fig. 2

where $\Delta\alpha_{3/4} = 1.6_{\text{deg}}$ for $C_2 = 0.2266$ and $k = 0.9$ from Eq. (13). Consequently if $\beta_{f,\chi}$ is evaluated at a higher velocity given by 1.1 V_c with ω_c constant, then Eq. (13) gives the increase in $\Delta\alpha_{f,\chi}$ when $\beta_{f,\chi}(1.1V_c/\omega_c)$ operates at the original (V_c/ω_c) with $k = (1.1)^{-1}$ and $C_2(1.1V_c/\omega_c) = 1.1\ (0.2266) = 0.2493$ therefore,

$$\alpha_{f,\chi}(V_c/\omega_c) = \tan^{-1}\left(\frac{\chi(1-k)(0.2493)}{\chi^2 + k(0.2493)^2}\right) \quad \text{with} \quad k = \left(\frac{1}{1.1}\right)$$

$$\alpha_{f,1} = 1.23_{\text{deg}}, \qquad \alpha_{f,\frac{3}{4}} = 1.57_{\text{deg}}$$

$$\alpha_{f,\frac{1}{2}} = 2.12_{\text{deg}}, \qquad \alpha_{f,\frac{1}{4}} = 2.73_{\text{deg}}$$

The fixed blade angles are given by,

$$\beta_{f,\chi} = \phi_\chi(1.1V_c/\omega_c) = \tan^{-1}(0.2493/\chi)$$

$$\alpha_{f,\chi}(1.1V_c/\omega_c) = 0 \tag{16}$$

Therefore either method is satisfactory, but Eq. (13) with $k = 1.1$–1.2 is the simplest method for determining the $\beta_{f,\chi} = \phi_\chi(kV_c/\omega_c)$ so that $\alpha_{f,\chi}(kV_c/\omega_c) = 0$ and at the design cruise speed (56 m/s) $\alpha_\chi(V_c/\omega_c) \approx \alpha_*$. Consequently in a steady shallow dive where V has increased more than, ω, so that $(V/\omega) = k(V_c/\omega_c)$, the interesting experimental situation can occur where,

$$\alpha_{f,\chi}(kV_c/\omega_c) = 0, \qquad \alpha_\chi(kV_c/\omega_c) = |\alpha_{f,\chi}(L=0)| \approx 44(t/c)_\chi \tag{17}$$

The required increase in blade setting for the maximum velocity (V_M) of a controllable-pitch propeller is also best determined by Eq. (13) because it gives $\Delta\alpha_\chi(V_M/\omega_M)$ for any change in $k = (V\omega_p/V_p\omega)$ for any given $\beta_\chi(V_p/\omega_p)$, measured from any fixed reference line (Fig. 1) because

$$-\Delta\alpha_\chi(kC_2) = \phi_\chi(kC_2) - \phi_\chi(C_2) = \tan^{-1}\left(\frac{\chi(k-1)C_2}{\chi^2 + kC_2^2}\right) \tag{18}$$

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